2.5
#20

$$3(1+t^2)\frac{dy}{dt} = 2ty(y^3-1)$$

$$3(1+t^2)\frac{dy}{dt} = 2ty^4-2ty$$

$$3(1+t^2)\frac{dy}{dt}+2ty = 2ty^4$$

$n=4$

$v=y^{1-4}=y^{-3}$ OR $y=v^{-\frac{1}{3}}$

$$\frac{dy}{dt}+\frac{2t}{3(1+t^2)}y = \frac{2t}{3(1+t^2)}y^4$$

$$\frac{dv}{dt}=\frac{dy^{-3}}{dy}\frac{dy}{dt}$$

$$-3y^{-4}\left(-\frac{1}{3}y^4\frac{dv}{dt}+\frac{2t}{3(1+t^2)}y\right)=\left(\frac{2t}{3(1+t^2)}y^4\right)(-3y^{-4})$$

$$\frac{dv}{dt}=\underline{-3y^{-4}}\frac{dy}{dt}$$

$$\frac{dy}{dt}=-\frac{1}{3}y^4\frac{dv}{dt}$$

$$\frac{dv}{dt}-\frac{2t}{1+t^2}y^{-3} = -\frac{2t}{1+t^2}$$

MANDATORY CHECKPOINT:

$$\frac{dv}{dt}-\frac{2t}{1+t^2}v = -\frac{2t}{1+t^2} \leftarrow \text{LINEAR}$$

$$\mu = e^{\int -\frac{2t}{1+t^2}dt} = e^{\int -\frac{du}{u}} = e^{-\ln|u|} = \frac{1}{|u|} = \frac{1}{1+t^2} \quad \ldots$$

$u=1+t^2$

$du=2t\,dt$

$$\frac{1}{1+t^2}\left(\frac{dv}{dt} - \frac{2t}{1+t^2}v\right) = \frac{1}{1+t^2}\left(-\frac{2t}{1+t^2}\right)$$

$$\frac{1}{1+t^2}\frac{dv}{dt} - \frac{2t}{(1+t^2)^2}v = -\frac{2t}{(1+t^2)^2}$$

MANDATORY CHECKPOI

$$\frac{d}{dt}(1+t^2)^{-1}$$

$$= -1(1+t^2)^{-2}(2t)$$

$$= -\frac{2t}{(1+t^2)^2} \checkmark$$

$$\frac{d}{dt}\left(\frac{1}{1+t^2}v\right) = -\frac{2t}{(1+t^2)^2}$$

$$\frac{1}{1+t^2}v = -\int \frac{2t}{(1+t^2)^2}dt$$

$$u = 1+t^2$$
$$du = 2t\,dt$$

$$-\int \frac{du}{u^2} = u^{-1} \text{ or } \frac{1}{u}$$

$$\frac{1}{1+t^2}v = \frac{1}{1+t^2} + C$$

$$v = 1 + C(1+t^2)$$

$$y^{-3} = 1 + C(1+t^2)$$

$$y = \frac{1}{\sqrt[3]{1+C(1+t^2)}}$$

$$1 + C(1+t^2)$$

$$= \underbrace{(C+1)}_{} + \underbrace{C}_{}t^2$$

$$\cancel{= K + Ct^2}$$

HIDDEN REL'N BETWEEN DIFFERENT COEF'S

4.1 LINEAR 2ND ORDER ODE

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = b(x)$$

$$y'' + \frac{a_1(x)}{a_2(x)}\,y' + \frac{a_0(x)}{a_2(x)}\,y = \frac{b(x)}{a_2(x)}$$

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

IF p, q ARE CONSTANT FUNCTIONS (IE. $p(x)=c$, $q(x)=k$), THE DE HAS CONSTANT COEFFICIENTS

IF NOT (EITHER $p(x) \neq c$ OR $q(x) \neq k$ OR BOTH), THE DE HAS VARIABLE COEFFICIENTS

IF $g = 0$ (IE. $g(x)=0$), THE DE IS HOMOGENEOUS

IF NOT (IE. $g(x) \neq 0$), THE DE IS NON-HOMOGENEOUS

$$y''+p(x)y'+q(x)y=g(x)$$

$$L[y]=y''+p(x)y'+q(x)y$$

$L[y(x)]$ (DIFFERENTIAL OPERATOR)

(FUNCTION WHOSE INPUT + OUTPUT ARE BOTH FUNCTIONS)

IE. DE CAN NOW BE WRITTEN AS $L[y]=g(x)$

$$L[y]=y''+py'+qy$$

WHERE p, q, y ARE FUNCTIONS OF SAME I.V.

IF $L[y]=y''-xy'+2y$

FIND $L[x^3]$

$$=(x^3)''-x(x^3)'+2(x^3)$$

$$=(6x)-x(3x^2)+2x^3$$

$$=6x-3x^3+2x^3$$

$$=6x-x^3$$

NOTE:

$y=x^3$ IS A SOL'N OF

$$y''-xy'+2y=6x-x^3$$

IF $L[y] = y'' + py' + qy$

THEN $L[y_1+y_2] = (y_1+y_2)'' + p(y_1+y_2)' + q(y_1+y_2)$

$$= (y_1''+y_2'') + p(y_1'+y_2') + qy_1 + qy_2$$

$$= \cancel{(}y_1''+y_2''\cancel{)} + py_1' + py_2' + qy_1 + qy_2$$

$$= (y_1''+py_1'+qy_1) + (y_2''+py_2'+qy_2)$$

$$\boxed{L[y_1+y_2] = L[y_1] + L[y_2]}$$

AND $L[cy] = (cy)'' + p(cy)' + q(cy)$

$c \in \mathbb{R}$ $\quad = (cy'') + p(cy') + cqy$

$$= c(y'' + py' + qy)$$

$$\boxed{L[cy] = c\,L[y]}$$

WE SAY L IS LINEAR

IE. L IS A LINEAR DIFFERENTIAL OPERATOR (LDO)

IS $L[y] = yy' + 3y''$ A LINEAR DIFF OPER (LDO)?

IE. $L[y_1 + y_2] = L[y_1] + L[y_2]$

AND

$L[cy] = cL[y]$?

CONSIDER $y = x^2$, $c = 2$

$$L[cy] = L[2x^2] = (2x^2)(2x^2)' + 3(2x^2)''$$
$$= 2x^2 \cdot (4x) + 3(4)$$
$$= 8x^3 + 12$$

$$cL[y] = 2L[x^2] = 2(x^2(x^2)' + 3(x^2)'')$$
$$= 2(x^2(2x) + 3(2))$$
$$= 2(2x^3 + 6)$$
$$= 4x^3 + 12$$

$$L[cy] \neq cL[y]$$

$\therefore$ L IS NOT A LDO

IF $L[y] = y'' + py' + qy$ AND y_1, y_2 ARE BOTH SOL'NS OF HOMOGENEOUS LINEAR DE $\underbrace{y'' + py' + qy = 0}_{L[y]=0}$

THEN FOR ALL $c_1, c_2 \in \mathbb{R}$

$c_1 y_1 + c_2 y_2$ IS ALSO A SOL'N OF THE DE.

SINCE y_1 IS A SOL'N OF $y'' + py' + qy = 0$,

$y_1'' + py_1' + qy_1 = 0$ IE. $L[y_1] = 0$

ALSO, $L[y_2] = 0$

$$
\begin{aligned}
L[c_1 y_1 + c_2 y_2] &= L[c_1 y_1] + L[c_2 y_2] \\
&= c_1 L[y_1] + c_2 L[y_2] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}
$$

IE. $c_1 y_1 + c_2 y_2$ IS A SOL'N OF $y'' + py' + qy = 0$

CONSIDER $y'' - 3y' + 2y = 0$ ← y, y', y'' LOOK LIKE EACH OTHER EXCEPT POSSIBLY FOR CONSTANT MULTIPLES

GUESS $y = e^{rx}$ $\quad r \in \mathbb{R}$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

$$y'' - 3y' + 2y = r^2 e^{rx} - 3re^{rx} + 2e^{rx} = 0$$

$$(r^2 - 3r + 2)e^{rx} = 0$$

$$\underbrace{(r-1)(r-2)}_{=0}\underbrace{e^{rx}}_{\neq 0 \text{ FUNCTION}} = 0$$

$r = 1$ OR 2

$y = e^x$ OR e^{2x}